CSC 202 - Discrete Math for Computer Scientists II

Professor Shai Simonson

**Assignment 2 –Boolean Algebra and Applications to Computer Science**

A. Truth Tables and Boolean Functions

1. Section 1.1 – 10. (In Edition 7: 10 is 8).
2. Section 1.3 – 14, 18 (In Edition 7: 14 and 18 are 12 and 14.
3. How many different Boolean functions with *n* variables are there? For *n=*2*,* list all the functions and identify as many as you can by name.
4. Prove that *a* → *b* is equivalent to ¬*b* → ¬*a* using a truth table.
5. Prove that *a* → *b* is not equivalent to *b* → *a*.

Recall a set of Boolean operations is called *complete* if and only if every Boolean function can be written using combinations of the operators. For example, the set {and, or, not} is complete because we can build any Boolean function from its truth table by focusing on the 1 outputs and using combinations of *and*s, *or*s, and *not*s. And, because of De Morgan’s laws, the sets {and, not}, and {or, not} are each complete.

1. Prove that *a*↑*b*, (*a* nand *b*), which is defined to be ¬(*a* ∧ *b*), is *complete*.
2. Write (*a*→*b*)→*b* using just ↑ (nand), then using just ↓ (nor).
3. It is natural to construct a disjunctive normal form (sum of products) formula from a truth table, by focusing on the outputs that are 1. Explain and show how to use a truth table in order to construct a conjunctive normal form (product of sums) for any Boolean formula *W*. Hint: Consider the disjunctive normal form for ¬*W*.

B. The exclusive-or operator ⊕, is defined by the rule that *a* ⊕ *b* is true whenever *a* or *b* is true but not both.

1. Calculate *x* ⊕ *x*, *x* ⊕ ¬*x, x* ⊕ 1, *x* ⊕ 0.
2. Prove or disprove that *x+* (*y* ⊕ *z*) *=* (*x+ y*) ⊕ (*x+ z*)
3. Prove or disprove that *x* ⊕ (*y* + *z*) *=* (*x* ⊕ *y*) + (*x* ⊕ *z*)
4. Write conjunctive normal form (product of sums) and disjunctive normal form (sum of products) formulae for *x* ⊕ *y*
5. The exclusive-or operator is not *complete.* Which one(s), if any, of the three operators {and, or, not} can be combined with exclusive-or to make a *complete* set.

C. Propositional Logic

Turn each statement about sets into propositional logic and manipulate the statements algebraically until they reduce to True. Recall that two sets A and B are equal, if and only if every element of A is in B and every element of B is in A.

1. The complement of the union of two sets equals the intersection of the complements.
2. The complement of the intersection of two sets equals the union of the complements.
3. (B −A) ∪ (C − A) = (B ∪ C) – A.
4. If two sets are subsets of each other then they are equal.

D. Sets Mimic Boolean Algebra

*A* ⊕ *B* is defined to be the set of all elements in *A* or *B* but not in both *A* and *B.*

1. Determine whether or not ⊕ is commutative. Prove your answer.
2. Determine whether ⊕ is associative. Prove your answer.
3. Determine whether ⊕ can be distributed over union. Prove your answer.
4. Determine whether ⊕ can be distributed over intersection. Prove your answer.

E. Logic Around Town

1. In the restroom of a fancy Italian restaurant in Mansfield, MA, there is a sign that reads: *Please do not leave valuables or laptop computers in your car.* Assuming that a laptop computer is considered a valuable, prove using formal logic, that the sentence *Please do not leave valuables in your car* is equivalent to the sign in the restroom. Prove that *Please do not leave laptops in your car* is not equivalent. Hint add “Laptop implies Valuable” to both sentences.
2. On bottles of Stonyfield Farms Organic Drinkable Yogurt, a statement reads: “ORGANIC means made without the use of antibiotics, hormones and toxic pesticides”. Explain what this claim means technically. According to this definition show that it would be possible for an organic drink to contain any one or two of these items. Assuming they were careless rather than malicious, how should they have written their claim?

F. First Order Logic

1. Section 1.4 – 26, 32, 58. (In Edition 7: 58 is 56). Section 1.5 – 28, 36.
2. Use logic with quantifiers and predicates to model the following three statements, using a universe of people: All students are taking classes. Some students are not motivated. Some people taking classes are not motivated.
3. Prove, using *resolution* methods, that the third statement follows logically from the first two. (Reminder: The method of resolution requires that you must take the conjunction of the first two statements and the negation of the third, and derive a contradiction. New clauses are created by *canceling* a positive and negative variable from each of two clauses, and taking the union of the remaining variables.)

G. Circuits and K-Maps

The following algebraic idea is central for Karnaugh maps. Karnaugh maps are a method of minimizing the size of circuits for digital logic design.

1. Using only algebraic manipulation, prove that the two Boolean formulae below are equivalent. (Hint: *x*(*a+*¬*a*) is equivalent to *x*.)

¬*yx* + ¬*zy* + ¬*xz* and ¬*xy* + ¬*yz* + ¬*zx*

1. Verify your results using a truth table.
2. Section 12.4 – 4, 6, 14.